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IMPROVED BOUNDS FOR AGGREGATED LINEAR PROGRAMS

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ABSTRACT

A method of Kallio for improving bounds on the optimal value of a linear program calculated from an intermediate iteration is used to improve Zipkin's bounds for an aggregated linear program. The method is also extended to obtain improved bounds for aggregated Markov decision problems, including improved bounds by dominance. Both theoretical and computational results are given, demonstrating the improvement due to these new bounds.

I. Introduction

One method for solving large scale linear programs (LPs) or large scale Markov Decision Processes (MDPs) is to aggregate the original LP into a smaller LP, solve the smaller LP, disaggregate the solution to the smaller LP back into the original tableau, and put bounds on the difference between the true optimum and this approximate optimum. Zipkin [5] has studied this problem extensively where the columns or rows or both are aggregated and disaggregated either by fixed weights, by "optimal" disaggregation, or else by dominance. For each of these problems, Zipkin gives upper and lower bounds for the value of a true optimum.

In related work, Kallio [2] derives bounds for a nonaggregated LP which is stopped at some iteration of the simplex method, using the primal and dual variables of the LP tableau at the stopped iteration. Then Kallio shows how to use marginal analysis to tighten the original bounds, using significantly less computation than would be required to perform one further iteration of the simplex method.

For a nonaggregated LP, Kallio's theorem 1 and Zipkin's proposition 2 are identical. Theorem 2.1 of this paper shows that Kallio's method of tightening the bounds extends readily to ~~an~~ aggregated LP. Theorem 3.1 demonstrates that Kallio's method also extends to discounted, infinite horizon MDPs that are approximately solved as aggregated LPs, and have error bounds developed by dominance, as in [3]. The advantage of these later bounds over the bounds in proposition 2 of [5] is that they require significantly less computation, though they produce a looser bound. The notation of this paper follows that of [5] whenever possible. Only weighted column aggregated problems are considered, as the extension to row and column aggregated problems are straightforward.

II. The Model

The original LP is:

$$\begin{aligned} z^* = \max\ & cx \\ \text{subject to } & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{2.1}$$

where $c = (c_j)$ is an n-vector, $b = (b_i)$ is an m-vector, $A = (a_{ij})$ is an $m \times n$ matrix, and $x = (x_j)$ is an n-vector of variables.

Let $\sigma = \{S_k : k = 1, \ldots, K\}$ be an arbitrary partition of $\{1, \ldots, n\}$, and $|S_k| = n_k$. Define $\mathcal{A}^k$ to be the submatrix of A consisting of those columns whose indices are in S_k. Define c^k and x^k similarly. Let g^k be a nonnegative n_k-vector whose components sum to unity, and define:

$$\bar{A}^k = \mathcal{A}^k g^k, \quad \bar{c}_k = c^k g^k \qquad k = 1, \ldots, K$$

Let $\bar{A} = \left(\bar{A}^1, \ldots, \bar{A}^K\right)$, $\bar{c} = \left(\bar{c}_1, \ldots, \bar{c}_K\right)$ and X a K-vector of variables. Then the weighted column aggregate problem of (2.1) is:

$$\begin{aligned} \bar{z} = \max\ & \bar{c}X \\ \text{subject to } & \bar{A}X \leq b \\ & X \geq 0 \end{aligned} \tag{2.2}$$

Zipkin [5] shows that for any partition $\sigma' = \{S'_k : k = 1, \ldots, K'\}$ of $\{1, \ldots, n\}$ such that there exists known positive numbers $\{d_1, \ldots, d_n\}$ and known nonnegative numbers $\{p_1, \ldots, p_K\}$ with:

$$\sum_{j \in S'_k} d_j x^*_j \leq p_k \qquad k = 1, \ldots, K' \tag{2.3}$$

then:

$$\bar{z} \leq z^* \leq \bar{z} + \sum_{k=1}^{K'} \left[\max_{j \in S'_k} \left\{ \left(c_j - \bar{u} A^j \right) / d_j \right\}^+ \right] p_k$$

where $\bar{u}$ is the vector of optimal dual variables of (2.2).

Kallio assumes there exists two n-vectors, ℓ and p, $0 \leq \ell \leq u < \infty$ such that if the constraint

$$\ell \leq x \leq p$$

is added to (2.1), then the value of an optimal solution is unchanged. Let $\bar{z}$ be the value of a current solution, let $\bar{u}$ be the present dual vector, and define $\bar{c}$, $\bar{\delta}$, and $\bar{\mu}$ by:

$$\bar{c} = (\bar{c}_j), \; \bar{c}_j = c_j - \bar{u}\, a_j \; ; \; \bar{\delta} = (\bar{\delta}_j), \; \bar{\delta}_j = \max\{0, \bar{c}_j\}$$

$$\bar{\mu} = (\bar{\mu}_j), \; -\bar{\mu}_j = \min\{0, \bar{c}_j\}$$

Then Kallio shows:

$$\bar{z} \leq z^* \leq \bar{u}\, b + \bar{\delta} p - \bar{\mu} \ell \tag{2.4}$$

Furthermore, the dual problem:

$$\begin{aligned} &\min\ ub + \delta p - \mu \ell \\ &\text{subject to } uA + \delta - \mu \geq c \\ &\qquad \delta, \mu \geq 0 \end{aligned} \tag{2.5}$$

restricted to $\{u | u = g + \theta d,\ \theta \varepsilon R\}$ can be solved by solving:

$$\begin{aligned} &\min\ K + \omega\theta + \delta p - \mu\ell \\ &\text{subject to } h\theta + \delta - \mu \geq c^{*} \\ &\qquad \delta,\ \mu \geq 0 \end{aligned} \tag{2.6}$$

$$K \equiv gb,\ \omega \equiv db$$

$$h \equiv (h_j) \equiv dA,\ c^{*} = (c_j^{*}) \equiv c - gA$$

Kallio shows how the dual problem (2.6) can be solved by marginal analysis, and yields a tighter bound than does (2.4). In particular, convenient values of d and g are $g \equiv 0$ and $d = \bar{u}$. Then $h_j = c_j - \bar{c}_j$, and $c_j^{*} = c_j$.

For the aggregated problem (2.2), suppose $\bar{z}$ is given, and also known are the optimal dual variables $\bar{u}$. Also assume (2.3), where for convenience it is assumed that $d_j = 1$ for all j. Consider the restricted primal to (2.1):

$$\begin{aligned} &\text{maximize } cx \\ &\quad \text{s.t. } Ax \leq b \\ &\qquad \sum_{j \varepsilon S_k} x_j \leq p_k \qquad k = 1, \ldots, K \\ &\qquad\quad x_j \geq 0 \end{aligned} \tag{2.7}$$

and the equivalent restricted dual:

$$\text{minimize } ub + \sum_{k=1}^{K} \delta_k p_k \tag{2.8}$$

$$\text{s.t. } \sum_i a_{ij} u_i + \delta_k \geq c_j \qquad j \in S_k;$$

$$j = 1, \ldots, m$$

$$u \geq 0$$

$$\delta \geq 0$$

Define $\bar{c}$ as before, but define $\bar{\delta} = \{\bar{\delta}_k\}$ by:

$$\bar{\delta}_k = \max_{j \in S_k} (\bar{c}_j)^+$$

By assumption (2.1) and (2.7) have the same optimal value z^*.

<u>Lemma 2.1</u> $(\bar{u}, \bar{\delta})$ <u>is a feasible solution to (2.8)</u>.

<u>Proof</u>. By definition $\bar{\delta}_k = \left(\max_{j \in S_k}\left(c_j - \sum_i a_{ij} \bar{u}_i\right)\right)^+ \geq \left(c_j - \sum_i a_{ij} \bar{u}_i\right)$

for any $j \in S_k$. Then each constraint in (2.8) satisfies:

$$\sum_i a_{ij} \bar{u}_i + \bar{\delta}_k \geq \sum_i a_{ij} \bar{u}_i + \left(c_j - \sum_i a_{ij} \bar{u}_i\right)^+ \geq c_j$$

since if $(c_j - \sum_i a_{ij} \bar{u}_i)$ is nonpositive, than $\sum_i a_{ij} \bar{u}_i \geq c_j$, so the constraint is feasible. If $(c_j - \sum_i a_{ij} \bar{u}_i)$ is positive, then the constraint is $c_j \geq c_j$, which again is feasible.

□

Lemma 2.1 leads to the extension of Kallio's theorem 1 for the aggregated LP. Consider (2.8) with u restricted to the set:

$$\{u \mid u = \theta\bar{u},\ \theta \in R\} \tag{2.9}$$

The new restricted dual is:

$$\begin{aligned} &\text{minimize } \bar{z}\theta + \sum_k \delta_k p_k \\ &\text{s.t. } \Big(\sum_i a_{ij}\bar{u}_i\Big)\theta + \delta_k \geq c_j \qquad j \in S_k \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad j = 1, \ldots, n \\ &\qquad\qquad\qquad u \geq 0 \\ &\qquad\qquad\qquad \delta_k \geq 0 \qquad k = 1, \ldots, K \end{aligned} \tag{2.10}$$

Let $z(\theta)$ be an optimal value of (2.10) for a given value of θ. Then an optimal solution to (2.8) with the added constraint (2.9) can be found by minimizing $z(\theta)$ with respect to θ, as in [2].

<u>Theorem 2.1.</u> <u>$z(\theta)$ is a convex and piecewise linear function. Let $z_-(\theta)$ be the left-hand derivative of $z(\theta)$ with respect to θ. Then the possible discontinuity points of $z_-(\theta)$ where an optimum can occur are at $\theta_1, \ldots, \theta_n$ where:</u>

$$\theta_j = \begin{cases} c_j \Big/ \sum_i a_{ij}\bar{u}_i & \sum_i a_{ij}\bar{u}_i \neq 0 \\ \infty & \sum_i a_{ij}\bar{u}_i = 0, \text{ for all } j. \end{cases}$$

Proof. For a fixed value of θ, (2.10) is decomposable into the following K subproblems:

$$\begin{aligned} &\text{minimize } \delta_k p_k \\ &\quad \text{s.t. } \delta_j \geq c_j - \theta\left(\sum_i a_{ij}\bar{u}_i\right) \qquad j \in S_k \end{aligned} \tag{2.11}$$

The solution of (2.11) is readily obtained as:

$$\max_{j \in S_k}\left(c_j - \theta\left(\sum_i a_{ij}\bar{u}_i\right)\right)^+ p_k \tag{2.12}$$

Possible points of discontinuity are at $c_j/\sum_i a_{ij}\bar{u}_i$ for all $j \in S_k$, and also at all values of θ such that

$$\left(c_j - \theta\left(\sum_i a_{ij}\bar{u}_i\right)\right) = \left(c_r - \theta\left(\sum_i a_{ir}\bar{u}_i\right)\right)$$

for $j, r \in S_k$. However, unless all such constraints in (2.11) are nonpositive, an optimum can only occur at the points $c_j/(\sum_i a_{ij}\bar{u}_i)$. Equation (2.12) is clearly convex and piecewise linear in θ. Then:

$$z(\theta) = \bar{z}\theta + \sum_{k=1}^{K}\left[\max_{j \in S_k}\left(c_j - \left(\theta \sum_i a_{ij}\bar{u}_i\right)\right)^+\right]p_k$$

which is convex and piecewise linear since it is the sum of such functions. Similarly, its points of discontinuity at an optimum can only be $\theta_1, \ldots, \theta_n$.

□

Let $z(\theta^*)$ be the minimum value of $z(\theta)$, that is an optimal solution to (2.8) subject to (2.9).

$$\text{Let } \xi_a = \sum_{k=1}^{K} \left[\max_{j \in S_k} \left\{c_j - \bar{u}A^j\right\}^+\right] p_k$$

which is the error term in Zipkin's bound.

<u>Corollary 2.1</u> $\bar{z} \leq z^* \leq z(\theta^*) \leq \bar{z} + \xi_a$

<u>Proof</u>. $z(\theta^*) \leq \bar{z} + \xi_a$ since $\bar{z} + \xi_a$ is equivalent to $z(1)$, and θ^* minimizes $z(\theta^*)$. By weak duality, a solution to (2.8) is greater than or equal to a solution to (2.7). Since (2.9) restricts (2.8), $z(\theta^*)$ is no less than an optimal value to (2.8), hence $z(\theta^*)$ is a legitimate upper bound.

□

Following and extending Kallio, the following method of marginal analysis yields an optimal value of θ. For any value of θ, define $J(\theta, k)$ as the j such that:

$$c_{j(\theta,k)} - \theta \sum_i a_{ij(\theta,k)} \bar{u}_i = \max_{j \in S_k} \left(c_j - \theta \sum_i a_{ij} \bar{u}_i\right)$$

and let $J(\theta) = \{j \mid j \in J(\theta, k),\ k = 1, \ldots, K\}$.

Then $z\_(\theta) = \bar{z} - \sum_{j \in I(\theta)} p_k \left(\sum_i a_{ij} \bar{u}_i\right)$

where $I(\theta) = \left\{j \mid j \in J(\theta);\ \theta_j \leq \theta \text{ and } \sum_i a_{ij} \bar{u}_i < 0 \text{ or } \theta_j > \theta \text{ and } \sum_i a_{ij} \bar{u}_i \geq 0\right\}$.

The marginal analysis is as follows:

1) Choose any $\theta_i \varepsilon \{\theta_1, \ldots, \theta_n\}$

2) Evaluate $z\_(\theta_i)$

3) If $z\_(\theta_i)$ equals zero or changes sign at θ_i, θ_i is optimal. Otherwise, if $z^*\_(\theta_i)$ is negative, increase to the next largest element in $\{\theta_1, \ldots, \theta_n\}$, and if $z\_(\theta_i)$ is positive, decrease to the next smallest element in $\{\theta_1, \ldots, \theta_n\}$.

At θ^*, the value of $z(\theta^*)$ can also be evaluated as:

$$z(\theta^*) = \bar{z}\theta + \sum_{j \varepsilon I(\theta)} \left(c_j - \theta \sum_i a_{ij} \bar{u}_i\right) p_k \qquad (2.13)$$

The summation in (2.13) contains at most K terms.

Example 1. This example is from Zipkin [5]. The original problem is:

$$\begin{aligned} z^* = \max \quad & 2.5x_1 + 3x_2 + 4x_3 + 5x_4 \\ \text{subject to} \quad & 4x_1 + 5x_2 + 7x_3 + 10x_4 \leq 54 \\ & x_1 + 2x_2 + x_3 + 2x_4 \leq 10 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

An optimal solution is $x_1^* = 16/3$, $x_3^* = 14/3$, $x_2^* = x_4^* = 0$, $z^* = 32$.

Let $K = 2$, $s_1 = \{1, 2\}$, $s_2 = \{3, 4\}$, and the column weights in each partition be (0.5, 0.5). Then $\bar{z} = 28\frac{5}{6}$ and $\bar{u} = \left(21/48, 25/48\right)$. Zipkin gives bounds for $p_1 = 10$, $p_2 = 8$:

$$28\tfrac{5}{6} \leq z^* \leq 34\tfrac{11}{24}$$

For finding the improved upper bound,

$$\theta_1 = 1.101 \quad \theta_2 = 0.4290; \quad \theta_3 = 1.116 \quad \theta_4 = 0.9231$$

$$\theta^* = \theta_1 = 1.101$$

and the improved bound is:

$$28\tfrac{5}{6} \le z^* \le 32.1855$$

which is an extremely tight upper bound on the true optimal value $z^* = 32$.

[illegible] [5], this is the same problem as example [illegible], but the aggregation weights are changed to (0.75, 0.25). Then z = [illegible], and u = ([illegible], [illegible]). Zipkin's bounds are

$$30\tfrac{2}{7} \le z^* \le 33\tfrac{23}{70}$$

For the improved bound:

$$\theta_1 = 1.051 \quad \theta_2 = 0.8917; \quad \theta_3 = 1.0606 \quad \theta_4 = 0.8794$$

$$\theta^* = \theta_1 = 1.051$$

and the improved bound is:

$$30\tfrac{2}{7} \le z^* \le 32.1231$$

which again is an extremely tight upper bound.

III. <u>Application to MDP's</u>

A discounted MDP defined on the integers $X = \{1, \ldots, n\}$ is controlled over an infinite planning horizon. For each state $x \in X$, feasible decisions $Y(x)$ are available. If $y \in Y(x)$ is the decision chosen, then an immediate reward $G(x, y)$ is received, and a transition is made to state $j \in X$ with probability P^y_{xj}. The one-period rewards are discounted by a factor α, $0 \leq \alpha < 1$, and the object is to maximize the total expected reward over an infinite horizon. It is well known that this problem can be solved by the following LP (d'Epenoux [1]):

$$
\begin{aligned}
&\text{maximize} \sum_{x=1}^{n} G(x, y)\, u^y_x \\
&\text{s.t.} \sum_{x=1}^{n} \sum_{y \in Y(x)} \left(\delta_{xj} - \alpha P^y_{xj}\right) u^y_x = 1 \qquad j = 1, \ldots, n \\
&\qquad\qquad u^y_x \geq 0 \qquad x \in X,\ y \in Y(x)
\end{aligned}
\tag{3.1}
$$

If it is assumed that $G(x, y)$ is uniformly (finitely) bounded from below, then the equalities in (3.1) can be transformed into inequalities.

MDPs that arise from real problems can be so large that both (3.1) and any of the bounds cited previously for an aggregated version of (3.1) are computationally infeasible. In [3], bounds are developed that are looser than C_k but require less computation. (Again, only column aggregation is considered, as improved lower bounds can be obtained similarly; see [4].) For each S_k, let $(x, y) \in S_k$ denote that the column associated with choosing action y from state x is in the partition S_k. Define:

$$G^k(x, y) \geq \max_{(x,\, y) \in S_k} G(x, y) \qquad k = 1, \ldots, K$$

$$P^k_{xj} \geq \max_{(x,\, y) \in S_k} P^y_{xj} \qquad \text{for all } j \in X$$

and for each k. Let $H_k \subset X$ be the set of x such that at least one $(x, y) \in S_k$. Let $\bar{f}$ be the optimal dual variables to an aggregated version of (3.1), then define f^*_k as:

$$f^*_k = \underset{x \in H_k}{\text{minimum}}\ \bar{f}_x \qquad k = 1, \ldots, K$$

Then the following bound on an aggregated LP can be derived [3]:

$$\bar{z} \leq z^* \leq \sum_{k=1}^{K} \left[\left(G^k(x, y) + \alpha \Sigma P^k_{xj}\, \bar{f}_j - f^*_k\right)^+\right] p_k \qquad (3.2)$$

The advantage of (3.2) is that only K terms are evaluated, that is only as many terms as are in the aggregated version of (3.1).

For each partition S_k, $k = 1, \ldots, K$ define two terms:

$$h^+_k \geq \max_{(x,\, y) \in S_k} \left\{\bar{f}_x - \alpha \Sigma P^y_{xj}\, \bar{f}_j\right\}$$

$$h^-_k \leq \min_{(x,\, y) \in S_k} \left\{\bar{f}_x - \alpha \Sigma P^y_{xj}\, \bar{f}_j\right\}$$

and define θ^+_k, θ^-_k as:

$$\theta^+_k = G^k(x, y)/h^+_k$$

$$\theta^-_k = G^k(x, y)/h^-_k$$

to θ to the value if the next smallest value of $\left\{\theta_1^+, \theta_1^-, \ldots, \theta_k^+, \theta_k^-\right\}$.

4) Evaluate $v(\theta)$ at θ^*.

Let θ^1 be an optimal θ found by this procedure, and θ^2 an optimal θ found by the procedure of section 2.

<u>Theorem 3.1</u> $\bar{z} \leq z^* \leq z(\theta^2) \leq v(\theta^1) \leq$ upper bound in (3.2).

<u>Proof</u>: From (2.11), the contribution of any partition S_k to $z(\theta)$ for fixed θ is:

$$\max_{(x,\, y) \in S_k} \left(G(x,\, y) - \theta(\bar{f}_x - \alpha \sum_j P^y_{xj}\, \bar{f}_j)\right)^+ p_k$$

$$\leq \max_{(x,\, y) \in S_k} \left(G^k(x,\, y) - \theta(\bar{f}_x - \alpha \sum_j P^y_{xj}\, \bar{f}_j)\right)^+ p_k$$

For $\theta \geq 0$:

$$\max_{(x,\, y) \in S_k} \left(G^k(x,\, y) - \theta(\bar{f}_x - \alpha \sum_j P^y_{xj}\, \bar{f}_j)\right)^+ \leq \left(G^k(x,\, y) - \theta h_k^-\right)^+ \tag{3.3a}$$

For $\theta < 0$:

$$\max_{(x,\, y) \in S_k} \left(G^k(x,\, y) - \theta(\bar{f}_x - \alpha \sum_j P^y_{xj}\, \bar{f}_j)\right)^+ \leq \left(G^k(x,\, y) - \theta h_k^+\right)^+ \tag{3.3b}$$

Equation (3.3a) and (3.3b) imply that the contribution of partition S_k to $v(\theta)$ is greater than or equal to the contribution of partition S_k to $z(\theta)$. This implies at an optimum:

$$z(\theta^2) \leq v(\theta^1)$$

The rest of the theorem follows from theorem 2.1, and the fact that the upper bound in (3.2) is exactly v(1).

□

Theorem 3.1 allows an improved bound to be calculated in much less computation than is needed for one simplex iteration on a problem with 2K columns.

Example 3. Mathews [6] gives the following equation for the return of salmon in the Naknek river:

$$x_{t+1} = 6.727\, e^{d}\, y_t \exp\{-0.859\, y_t\} \tag{3.4}$$

where d is distributed as N(0, 0.1444). Equation (3.4) is discretized on the set:

$$X = \{0,\ 0.125,\ 0.25,\ 0.375,\ 0.5,\ 0.625,\ 0.75,\ 0.875,\ 1,\ 1.125\ \ 1.25,\ 1.375,\ 1.5,\ 1.625,\ 1.75,\ 1.875,\ 2,\ 2.5,\ 3,\ 3.5,\ 4,\ 4.5,\ 5,\ 5.5,\ 6,\ 6.5,\ 7,\ 7.5,\ 8,\ 8.5,\ 9\}$$

where the units are 10^6 fish, and the decision set is:

$$Y(x) = \{y : 0 \le y \le x,\ x,\ y \in X\} \cdot$$

A forthcoming paper will discuss in greater detail the discretization procedure and policy implications.

The one-period reward is (x - y), and the discount factor used is $\alpha = 0.97$. The true optimal value is $z^* = 1918.1672$, by following a base stock policy given by:

$$y = \min\{x,\ 0.75\} \quad .$$

The partitions, aggregating weights, and p_k, P^k, and G^k are given in Table 3.1. The bounds using ξ_a are:

$$624.2476 \leq z^* \leq 14{,}849.0491$$

The improved bounds using theorem 2.1 are:

$$624.2476 \leq z^* \leq 9{,}051.6241$$

which occurs at $\theta^* = (-0.027)$. This comes from partition 2, from the column associated with $u^{0.25}_{0.75}$.

The solution to the aggregated MDP gives a randomized policy for all states, weighted between $y = \{0, 0.125, 0.25, 0.375, 0.5\}$, which is not a desirable solution. Three observations arise from these results:

1) The improved upper bound gives a better idea of the true optimal value.
2) The unimproved upper bound gives a better feel of how "good" the present partitioning and aggregation is--that is the larger spread is more indicative that implementing the disaggregated "optimal" policy is probably undesirable.
3) The problem is highly deterministic. The root of $\frac{d}{dx}\left(\alpha\, 6.727\, y \exp(-0.859\, y) - y\right) = 0$ for $\alpha = 0.97$ is 0.774. Rounded down to the nearest grid point yields the proper optimal policy.

For this problem, the bounds by dominance are:

$$624.2476 \leq z^* \leq 347,663.366 \tag{3.5}$$

which are not very good, though simple to calculate.

A simple improved bound by dominance can be found at $\theta = 0$, though this is not the best improvement. In this case, the upper bound is:

$$\sum_{k=1}^{K} [G^k(x, y)]^+ p_k$$

which gives bounds of

$$624.2476 \leq z^* \leq 9,147.0000$$

This is a dramatic improvement over (3.5), and almost as good as the best upper bound. For very large MDP's, bounds by dominance may be the only ones available, and the improvement afforded by theorem 3.1 and the tightness of the bounds are encouraging.

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